

# Quantum Lie algebras via modified Reflection Equation Algebra

Dimitri Gurevich\*

*USTV, Université de Valenciennes, 59304 Valenciennes, France*

Pavel Saponov†

*Division of Theoretical Physics, IHEP, 142284 Protvino, Russia*

## 1 Introduction

A Lie super-algebra was historically the first generalization of the notion of a Lie algebra. Lie super-algebras were introduced by physicists in studying dynamical models with fermions. In contrast with the usual Lie algebras defined via the classical flip  $P$  interchanging any two elements  $P(X \otimes Y) = Y \otimes X$ , the definition of a Lie super-algebra is essentially based on a super-analog of the permutation  $P$ . This super-analog is defined on a  $\mathbb{Z}_2$ -graded vector space  $V = V_{\bar{0}} \oplus V_{\bar{1}}$  where  $\bar{0}, \bar{1} \in \mathbb{Z}_2$  is a "parity". On homogeneous elements (i.e. those belonging to either  $V_{\bar{0}}$  or  $V_{\bar{1}}$ ) its action is  $P(X \otimes Y) = (-1)^{\bar{X}\bar{Y}} Y \otimes X$ , where  $\bar{X}$  stands for the parity of a homogeneous element  $X \in V$ .

Then a Lie super-algebra is the following data

$$(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, P : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \otimes \mathfrak{g}, [\ , \ ] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}),$$

where  $\mathfrak{g}$  is a super-space,  $P$  is a super-flip, and  $[\ , \ ]$  is a Lie super-bracket, i.e. a linear operator which is subject to three axioms:

1.  $[X, Y] = -(-1)^{\bar{X}\bar{Y}}[Y, X]$ ;
2.  $[X, [Y, Z]] + (-1)^{\bar{X}(\bar{Y}+\bar{Z})}[Y, [Z, X]] + (-1)^{\bar{Z}(\bar{X}+\bar{Y})}[Z, [X, Y]] = 0$ ;
3.  $\overline{[X, Y]} = \bar{X} + \bar{Y}$ .

Here  $X, Y, Z$  are assumed to be arbitrary homogenous elements of  $\mathfrak{g}$ . Note that all axioms can be rewritten via the corresponding super-flip. For instance the axiom 3 takes the form

$$P(X \otimes [Y, Z]) = [\ , \ ]_{12} P_{23} P_{12}(X \otimes Y \otimes Z).$$

(As usual, the indices indicate the space(s) where a given operator is applied.)

---

\*gurevich@univ-valenciennes.fr

†Pavel.Saponov@ihep.ru

In this paper we discuss the problem what is a possible generalization of the notion of a Lie super-algebra related to "flips" of more general type.

The first generalization of the notion of a Lie super-algebra was related to gradings different from  $\mathbb{Z}_2$ . The corresponding Lie type algebras were called  $\Gamma$ -graded ones (cf. [Sh]).

The next step was done in [G1] where there was introduced a new generalization of the Lie algebra notion related to an involutive symmetry defined as follows. Let  $V$  be a vector space over a ground field  $\mathbb{K}$  (usually  $\mathbb{C}$  or  $\mathbb{R}$ ) and  $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$  be a linear operator. It is called a *braiding* if it satisfies the quantum Yang-Baxter equation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23},$$

where  $R_{12} = R \otimes I$ ,  $R_{23} = I \otimes R$  are operators in the space  $V^{\otimes 3}$ . If such a braiding satisfies the condition  $R^2 = I$  (resp.,  $(R - qI)(R + q^{-1}I) = 0$ ,  $q \in \mathbb{K}$ ) we call it an *involutive symmetry* (resp., a *Hecke symmetry*). In the latter case  $q$  is assumed to be generic<sup>1</sup>.

Two basic examples of *generalized* Lie algebras are analogs of the Lie algebras  $gl(n)$  and  $sl(n)$  (or of their super-analogs  $gl(m|n)$  and  $sl(m|n)$ ). They can be associated to any "skew-invertible" (see Section 2) involutive symmetry  $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ . We denote them  $gl(V_R)$  and  $sl(V_R)$  respectively. The generalized Lie algebras  $gl(V_R)$  and  $sl(V_R)$  are defined in the space  $\text{End}(V)$  of endomorphisms of the space  $V$ . Their enveloping algebras  $U(gl(V_R))$  and  $U(sl(V_R))$  (which can be defined in a natural way) are equipped with a braided Hopf structure such that the coproduct coming in its definition acts on the generators  $X \in gl(V_R)$  or  $sl(V_R)$  in the classical manner:  $\Delta : X \rightarrow X \otimes 1 + 1 \otimes X$ .

Moreover, if an involutive symmetry  $R$  is a deformation of the usual flip (or super-flip) the enveloping algebras  $U(gl(V_R))$  and  $U(sl(V_R))$  are deformations of their classical (or super-) counterparts.

There are known numerous attempts to define a quantum (braided) Lie algebra similar to generalized ones but without assuming  $R$  to be involutive. Let us mention some of them: [W], [LS], [DGG], [GM]. In this note we compare the objects defined there with  $gl$  type Lie algebras-like objects introduced recently in [GPS]. Note that the latter objects can be associated with any skew-invertible Hecke symmetry, in particular, that related to Quantum Groups (QG) of  $A_n$  series. Their enveloping algebras are treated in terms of the modified reflection equation algebra (mREA) defined below. These enveloping algebras have good deformation properties and the categories of their finite dimensional *equivariant* representations look like those of the Lie algebras  $gl(m|n)$ . Moreover, these algebras can be equipped with a structure of braided bi-algebras. Though the corresponding coproduct acts on the generators of the algebras in a non-classical way it is in a sense intrinsic (it has nothing in common with the coproduct in the QGs). Moreover, it allows to define braided analogs of (co)adjoint vectors fields.

We think that apart from generalized Lie algebras related to involutive symmetries (described in Section 2) there is no general definition of a quantum (braided) Lie algebra. Moreover, reasonable quantum Lie algebras exist only for the  $A_n$  series (or more generally, for any skew-invertible Hecke symmetry). As for the quantum Lie algebras of the

---

<sup>1</sup>Note that there exists a big family of Hecke and involutive symmetries which are not deformations of the usual flip (cf. [G2]). Even the Poincaré-Hilbert (PH) series corresponding to the "symmetric"  $\text{Sym}(V) = T(V)/\langle \mathfrak{S}(qI - R) \rangle$  and "skew-symmetric"  $\bigwedge(V) = T(V)/\langle \mathfrak{S}(q^{-1}I + R) \rangle$  algebras can drastically differ from the classical ones, whereas the PH series are stable under a deformation.

$B_n, C_n, D_n$  series introduced in [DGG], their enveloping algebras are not deformations of their classical counterparts and for this reason they are somewhat pointless objects.

**Acknowledgement** The work of P.Saponov was partially supported by RFBR grant no. 05-01-01086.

## 2 Generalized Lie algebras

Let  $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$  be an involutive symmetry. Then the data

$$(V, R, [\cdot, \cdot] : V^{\otimes 2} \rightarrow V)$$

is called a *generalized Lie algebra* if the following holds

1.  $[\cdot, \cdot] R(X \otimes Y) = -[X, Y]$ ;
2.  $[\cdot, \cdot][\cdot, \cdot]_{12}(I + R_{12}R_{23} + R_{23}R_{12})(X \otimes Y \otimes Z) = 0$ ;
3.  $R[\cdot, \cdot]_{12}(X \otimes Y \otimes Z) = [\cdot, \cdot]_{12}R_{23}R_{12}(X \otimes Y \otimes Z)$ .

Such a generalized Lie algebra is denoted  $\mathfrak{g}$ .

Note that the generalized Jacobi identity (the axiom 2) can be rewritten in one of the following equivalent forms

$$\begin{aligned} & [\cdot, \cdot][\cdot, \cdot]_{23}(I + R_{12}R_{23} + R_{23}R_{12})(X \otimes Y \otimes Z) = 0; \\ & [\cdot, \cdot][\cdot, \cdot]_{12}(X(Y \otimes Z - R(Y \otimes Z))) = [X, [Y, Z]]; \\ & [\cdot, \cdot][\cdot, \cdot]_{23}((X \otimes Y - R(X \otimes Y))Z) = [[X, Y], Z]. \end{aligned}$$

**Example 1** If  $R$  is the usual flip then the third axiom is fulfilled automatically and we get a usual Lie algebra. If  $R$  is a super-flip then we get a Lie super-algebra. In the both cases  $R$  is involutive.

The enveloping algebras of the generalized Lie algebra  $\mathfrak{g}$  can be defined in a natural way:

$$U(\mathfrak{g}) = T(V) / \langle X \otimes Y - R(X \otimes Y) - [X, Y] \rangle.$$

(Hereafter  $\langle I \rangle$  stands for the ideal generated by a set  $I$ .) Let us introduce the symmetric algebra  $\text{Sym}(\mathfrak{g})$  of the generalized Lie algebra  $\mathfrak{g}$  by the same formula but with 0 instead of the bracket in the denominator of the above formula.

For this algebra there exists a version of the Poincaré-Birhoff-Witt theorem.

**Theorem 2** *The algebra  $U(\mathfrak{g})$  is canonically isomorphic to  $\text{Sym}(\mathfrak{g})$ .*

A proof can be obtained via the Koszul property established in [G2] and the results of [PP]. Also, note that similarly to the classical case this isomorphism can be realized via a symmetric (w.r.t. the symmetry  $R$ ) basis.

**Definition 3** We say that a given braiding  $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$  is skew-invertible if there exists a morphism  $\Psi : V^{\otimes 2} \rightarrow V^{\otimes 2}$  such that

$$\text{Tr}_2 \Psi_{12} R_{23} = P_{13} = \text{Tr}_2 \Psi_{23} R_{12}$$

where  $P$  is the usual flip.

If  $R$  is a skew-invertible braiding, a "categorical significance" can be given to the dual space of  $V$ . Let  $V^*$  be the vector space dual to  $V$ . This means that there exist a non-degenerated pairing  $\langle , \rangle : V^* \otimes V \rightarrow \mathbb{K}$  and an extension of the symmetry  $R$  to the space  $(V^* \oplus V)^{\otimes 2} \rightarrow (V^* \oplus V)^{\otimes 2}$  (we keep the same notation for the extended braiding) such that the above pairing is  $R$ -invariant. This means that on the space  $V^* \otimes V \otimes W$  (resp.,  $W \otimes V^* \otimes V$ ) where either  $W = V$  or  $W = V^*$  the following relations hold

$$R \langle , \rangle_{12} = \langle , \rangle_{23} R_{12} R_{23} \quad (\text{resp., } R \langle , \rangle_{23} = \langle , \rangle_{12} R_{23} R_{12}).$$

(Here as usual, we identify  $X \in W$  with  $X \otimes 1$  and  $1 \otimes X$ .)

Note that if such an extension exists it is unique. By fixing bases  $x_i \in V$  and  $x_i \otimes x_j \in V^{\otimes 2}$  we can identify the operators  $R$  and  $\Psi$  with matrices  $\|R_{ij}^{kl}\|$  and  $\|\Psi_{ij}^{kl}\|$  respectively. For example,

$$R(x_i \otimes x_j) = R_{ij}^{kl} x_k \otimes x_l$$

(from now on we assume the summation over the repeated indices).

Then the above definition can be presented in the following matrix form

$$R_{ij}^{kl} \Psi_{lm}^{jn} = \delta_m^k \delta_i^n.$$

If  ${}^i x$  is the left dual basis of the space  $V^*$ , i.e. such that  $\langle {}^j x, x_i \rangle = \delta_i^j$  then we put

$$\langle x_i, {}^j x \rangle = \langle , \rangle \Psi_{ik}^{jl} x \otimes x_l = C_i^j, \quad \text{where } C_i^j = \Psi_{ik}^{jk}.$$

(Note that the operator  $\Psi$  is a part of the braiding  $R$  extended to the space  $(V^* \oplus V)^{\otimes 2}$ .) By doing so, we ensure  $R$ -invariance of the pairing  $V \otimes V^* \rightarrow \mathbb{K}$ .

As shown in [GPS] for any skew-invertible Hecke symmetry  $R$  the following holds

$$C_i^j B_j^k = q^{-2a} \delta_i^k, \quad \text{where } B_i^j = \Psi_{ki}^{kj}$$

with an integer  $a$  depending on the the HP series of the algebra  $\text{Sym}(V)$  (see footnote 1). So, if  $q \neq 0$  the operators  $C$  and  $B$  (represented by the matrices  $\|C_i^j\|$  and  $\|B_j^k\|$  respectively) are invertible. Therefore, we get a non-trivial pairing

$$\langle , \rangle : (V \oplus V^*)^{\otimes 2} \rightarrow \mathbb{K}$$

which is  $R$ -invariant.

Note that these operators  $B$  and  $C$  can be introduced without fixing any basis in the space  $V$  as follows

$$B_2 = \text{Tr}_{(1)}(\Psi_{12}), \quad C_1 = \text{Tr}_{(2)}(\Psi_{12}). \quad (1)$$

Let us exhibit an evident but very important property of these operators

$$\text{Tr}_{(1)}(B_1 R_{12}) = I, \quad \text{Tr}_{(2)}(C_2 R_{12}) = I. \quad (2)$$

By fixing the basis  $h_i^j = x_i \otimes x^j$  in the space  $\text{End}(V) \cong V \otimes V^*$  equipped with the usual product

$$\circ : \text{End}(V)^{\otimes 2} \rightarrow \text{End}(V)$$

we get the following multiplication table  $h_i^j \circ h_k^l = \delta_k^j h_i^l$ .

Below we use another basis in this algebra, namely that  $l_i^j = x_i \otimes x^j$  where  $x^j$  is the *right dual* basis in the space  $V^*$ , i.e. such that  $\langle x_i, x^j \rangle = \delta_i^j$ . Note that the multiplication table for the the product  $\circ$  in this basis is  $l_i^j \circ l_k^m = B_k^j l_i^m$  (also see formula (6)).

Let  $R$  be the above extension of a skew-invertible braiding to the space  $(V^* \oplus V)^{\otimes 2}$ . Then a braiding  $R_{\text{End}(V)} : \text{End}(V)^{\otimes 2} \rightarrow \text{End}(V)^{\otimes 2}$  can be defined in a natural way:

$$R_{\text{End}(V)} = R_{23} R_{34} R_{12} R_{23},$$

where we used the isomorphism  $\text{End}(V) \cong V \otimes V^*$ .

Observe that the product  $\circ$  in the space  $\text{End}(V)$  is  $R$ -invariant and therefore  $R_{\text{End}(V)}$ -invariant. Namely, we have

$$\begin{aligned} R_{\text{End}(V)}(X \circ Y, Z) &= \circ_{23}(R_{\text{End}(V)})_{12}(R_{\text{End}(V)})_{23}(X \otimes Y \otimes Z), \\ R_{\text{End}(V)}(X, Y \circ Z) &= \circ_{12}(R_{\text{End}(V)})_{23}(R_{\text{End}(V)})_{12}(X \otimes Y \otimes Z). \end{aligned}$$

**Example 4** Let  $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$  be a skew-invertible involutive symmetry. Define a generalized Lie bracket by the rule

$$[X, Y] = X \circ Y - \circ R_{\text{End}(V)}(X \otimes Y).$$

Then the data  $(\text{End}(V), R_{\text{End}(V)}, [\ , \ ])$  is a generalized Lie algebra (denoted  $gl(V_R)$ ).

Besides, define the  $R$ -trace  $\text{Tr}_R : \text{End}(V) \rightarrow \mathbb{K}$  as follows

$$\text{Tr}_R(h_i^j) = B_j^i h_i^j, \quad X \in \text{End}(V).$$

The  $R$ -trace possesses the following properties :

- The pairing

$$\text{End}(V) \otimes \text{End}(V) \rightarrow \mathbb{K} : X \otimes Y \mapsto \langle X, Y \rangle = \text{Tr}_R(X \circ Y)$$

is non-degenerated;

- It is  $R_{\text{End}(V)}$ -invariant in the following sense

$$\begin{aligned} R_{\text{End}(V)}((\text{Tr}_R X) \otimes Y) &= (I \otimes \text{Tr}_R) R_{\text{End}(V)}(X \otimes Y), \\ R_{\text{End}(V)}(X \otimes (\text{Tr}_R Y)) &= (\text{Tr}_R \otimes I) R_{\text{End}(V)}(X \otimes Y); \end{aligned}$$

- $\text{Tr}_R[\ , \ ] = 0$ .

Therefore the set  $\{X \in gl(V_R) | \text{Tr}_R X = 0\}$  is closed w.r.t. the above bracket. Moreover, this subspace squared is invariant w.r.t the symmetry  $R_{\text{End}(V)}$ . Therefore this subspace (denoted  $sl(V_R)$ ) is a generalized Lie subalgebra.

Observe that the enveloping algebra of any generalized Lie algebra possesses a braided Hopf algebra structure such that the coproduct  $\Delta$  and antipode  $S$  are defined on the generators in the classical way

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad S(X) = -X.$$

For details the reader is referred to [G2].

Also, observe that while  $R$  is a super-flip the generalized Lie algebra  $gl(V_R)$  (resp.,  $sl(V_R)$ ) is nothing but the Lie super-algebras  $gl(m|n)$  (resp.,  $sl(m|n)$ ).

### 3 Quantum Lie algebras for $B_n, C_n, D_n$ series

In this Section we restrict ourselves to the braidings coming from the QG  $U_q(\mathfrak{g})$  where  $\mathfrak{g}$  is a Lie algebra of one of the series  $B_n, C_n, D_n$ . By the Jacobi identity, the usual Lie bracket

$$[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$$

is a  $\mathfrak{g}$ -morphism.

Let us equip the space  $\mathfrak{g}$  with a  $U_q(\mathfrak{g})$  action which is a deformation of the usual adjoint one. The space  $\mathfrak{g}$  equipped with such an action is denoted  $\mathfrak{g}_q$ . Our immediate goal is to define an operator

$$[\cdot, \cdot]_q : \mathfrak{g}_q \otimes \mathfrak{g}_q \rightarrow \mathfrak{g}_q$$

which would be a  $U_q(\mathfrak{g})$ -covariant deformation of the initial Lie bracket. This means that the  $q$ -bracket satisfies the relation

$$[\cdot, \cdot]_q(a_1(X) \otimes a_2(Y)) = a([X \otimes Y]_q),$$

where  $a$  is an arbitrary element of the QG  $U_q(\mathfrak{g})$ ,  $a_1 \otimes a_2 = \Delta(a)$  is the Sweedler notation for the QG coproduct  $\Delta$ , and  $a(X)$  stands for the result of applying the element  $a \in U_q(\mathfrak{g})$  to an element  $X \in \mathfrak{g}_q$ .

Let us show that  $U_q(\mathfrak{g})$ -covariance of the bracket entails its  $R$ -invariance where  $R = P\pi_{\mathfrak{g} \otimes \mathfrak{g}}(\mathcal{R})$  is the image of the universal quantum  $R$ -matrix  $\mathcal{R}$  composed with the flip  $P$ . Indeed, due to the relation

$$\Delta_{12}(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23},$$

we have (by omitting  $\pi_{\mathfrak{g} \otimes \mathfrak{g}}$ )

$$\begin{aligned} R[\cdot, \cdot]_1(X \otimes Y \otimes Z) &= P\mathcal{R}([X, Y] \otimes Z) = P[\cdot, \cdot]_{12}\Delta_{12}\mathcal{R}(X \otimes Y \otimes Z) = \\ P[\cdot, \cdot]_{12}\mathcal{R}_{13}\mathcal{R}_{23}(X \otimes Y \otimes Z) &= P[\cdot, \cdot]_{12}P_{13}R_{13}P_{23}R_{23}(X \otimes Y \otimes Z) = \\ P[\cdot, \cdot]_{12}P_{13}P_{23}R_{12}R_{23}(X \otimes Y \otimes Z) &= [\cdot, \cdot]_{23}R_{12}R_{23}(X \otimes Y \otimes Z). \end{aligned}$$

Finally, we have

$$R[\cdot, \cdot]_{12} = [\cdot, \cdot]_{23}R_{12}R_{23}, \quad R[\cdot, \cdot]_{23} = [\cdot, \cdot]_{12}R_{23}R_{12}$$

(the second relation can be obtained in a similar way).

Thus, the  $U_q(\mathfrak{g})$ -covariance of the bracket  $[\cdot, \cdot]_q$  can be considered as an analog of the axiom 3 from the above list. In fact, if  $\mathfrak{g}$  belongs to one of the series  $B_n, C_n$  or  $D_n$ , this property suffices for unique (up to a factor) definition of the bracket  $[\cdot, \cdot]_q$ . Indeed, in this case it is known that if one extends the adjoint action of  $\mathfrak{g}$  to the space  $\mathfrak{g} \otimes \mathfrak{g}$  (via the coproduct in the enveloping algebra), then the latter space is multiplicity free with respect to this action. This means that there is no isomorphic irreducible  $\mathfrak{g}$ -modules in the space  $\mathfrak{g} \otimes \mathfrak{g}$ . In particular, the component isomorphic to  $\mathfrak{g}$  itself appears only in the skew-symmetric subspace of  $\mathfrak{g} \otimes \mathfrak{g}$ . A similar property is valid for decomposition of the space  $\mathfrak{g}_q \otimes \mathfrak{g}_q$  into a direct sum of irreducible  $U_q(\mathfrak{g})$ -modules (recall that  $q$  is assumed to be generic).

Thus, the map  $[\cdot, \cdot]_q$ , being a  $U_q(\mathfrak{g})$ -morphism, must kill all components in the decomposition of  $\mathfrak{g}_q \otimes \mathfrak{g}_q$  into a direct sum of irreducible  $\mathfrak{g}_q$ -submodules except for the component isomorphic to  $\mathfrak{g}_q$ . Being restricted to this component, the map  $[\cdot, \cdot]_q$  is an isomorphism.

This property uniquely defines the map  $[\cdot, \cdot]_q$  (up to a non-zero factor). For an explicit computation of the structure constants of the  $q$ -bracket  $[\cdot, \cdot]_q$  the reader is referred to the paper [DGG]. Note that the authors of that paper embedded the space  $\mathfrak{g}_q$  in the QG  $U_q(\mathfrak{g})$ . Nevertheless, it is possible to do all the calculations without such an embedding but using the QG just as a substitute of the corresponding symmetry group.

Now, we want to define the enveloping algebra of a quantum Lie algebra  $\mathfrak{g}_q$ . Since the space  $\mathfrak{g}_q \otimes \mathfrak{g}_q$  is multiplicity free, we conclude that there exists a unique  $U_q(\mathfrak{g})$ -morphism  $P_q : \mathfrak{g}_q \otimes \mathfrak{g}_q \rightarrow \mathfrak{g}_q \otimes \mathfrak{g}_q$  which is a deformation of the usual flip and such that  $P_q^2 = I$ . Indeed, in order to introduce such an operator it suffices to define  $q$ -analogs of symmetric and skew-symmetric components in  $\mathfrak{g}_q \otimes \mathfrak{g}_q$ . Each of them can be defined as a direct sum of irreducible  $U_q(\mathfrak{g})$ -submodules of  $\mathfrak{g}_q \otimes \mathfrak{g}_q$  which are  $q$ -counterparts of the  $U_q(\mathfrak{g})$ -modules entering the usual symmetric and skew-symmetric subspaces respectively.

Now, the enveloping algebra can be defined as a quotient

$$U(\mathfrak{g}_q) = T(\mathfrak{g}_q) / \langle X \otimes Y - P_q(X \otimes Y) - [\cdot, \cdot]_q \rangle.$$

Thus, we have defined the quantum Lie algebra  $\mathfrak{g}_q$  and its enveloping algebra related to the QG of  $B_n, C_n, D_n$  series. However, the question what properties of these quantum Lie algebras are similar to those of generalized Lie algebras is somewhat pointless since the algebra  $U(\mathfrak{g}_q)$  is not a deformation of its classical counterpart. Moreover, its " $q$ -commutative" analog (which is defined similarly to the above quotient but without the  $q$ -bracket  $[\cdot, \cdot]_q$  in the denominator) is not a deformation of the algebra  $\text{Sym}(\mathfrak{g})$ . For the proof, it suffices to verify that the corresponding semiclassical term is not a Poisson bracket. (However, it becomes Poisson bracket upon restriction to the corresponding algebraic group.)

**Remark 5** A similar construction of a quantum Lie algebra is valid for any skew-invertible braiding of the Birman-Murakami-Wenzl type. But for the same reason it is out of our interest.

Also, note that the Lie algebra  $sl(2)$  possesses a property similar to that above: the space  $sl(2) \otimes sl(2)$  being equipped with the extended adjoint action is a multiplicity free  $sl(2)$ -module. So, the corresponding quantum Lie algebra and its enveloping algebra can be constructed via the same scheme. However, the latter algebra is a deformation of its classical counterpart. This case is considered in the next Sections as a part of our general construction related to Hecke symmetries.

## 4 Modified Reflection Equation Algebra and its representation theory

In this section we shortly describe the modified reflection equation algebra (mREA) and the quasitensor Schur-Weyl category of its finite dimensional *equivariant* representations. Our presentation is based on the work [GPS], where these objects were considered in full detail.

The starting point of all constructions is a Hecke symmetry  $R$ . As was mentioned in Introduction, the Hecke symmetry is a linear operator  $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ , satisfying the quantum Yang-Baxter equation and the additional Hecke condition

$$(R - qI)(R + q^{-1}I) = 0,$$

where a nonzero  $q \in \mathbb{K}$  is generic, in particular, is not a primitive root of unity. Besides, we assume  $R$  to be skew-invertible (see Definition 3).

Fixing bases  $x_i \in V$  and  $x_i \otimes x_j \in V^{\otimes 2}$ ,  $1 \leq i, j \leq N = \dim V$ , we identify  $R$  with a  $N^2 \times N^2$  matrix  $\|R_{ij}^{kl}\|$ . Namely, we have

$$R(x_i \otimes x_j) = R_{ij}^{kl} x_k \otimes x_l, \quad (3)$$

where the lower indices label the rows of the matrix, the upper ones — the columns.

As is known, the Hecke symmetry  $R$  allows to define a representations  $\rho_R$  of the  $A_{k-1}$  series Hecke algebras  $H_k(q)$ ,  $k \geq 2$ , in tensor powers  $V^{\otimes k}$ :

$$\rho_R : H_k(q) \rightarrow \text{End}(V^{\otimes k}) \quad \rho_R(\sigma_i) = R_i := I^{\otimes(i-1)} \otimes R \otimes I^{\otimes(k-i-1)},$$

where elements  $\sigma_i$ ,  $1 \leq i \leq k-1$  form the set of the standard generators of  $H_k(q)$ .

The Hecke algebra  $H_k(q)$  possesses the primitive idempotents  $e_a^\lambda \in H_k(q)$ , which are in one-to-one correspondence with the set of all standard Young tableaux  $(\lambda, a)$ , corresponding to all possible partitions  $\lambda \vdash k$ . The index  $a$  labels the tableaux of a given partition  $\lambda$  in accordance with some ordering.

Under the representation  $\rho_R$ , the primitive idempotents  $e_a^\lambda$  are mapped into the projection operators

$$E_a^\lambda(R) = \rho_R(e_a^\lambda) \in \text{End}(V^{\otimes k}), \quad (4)$$

these projectors being some polynomials in  $R_i$ ,  $1 \leq i \leq k-1$ .

Under the action of these projectors the spaces  $V^{\otimes k}$ ,  $k \geq 2$ , are expanded into the direct sum

$$V^{\otimes k} = \bigoplus_{\lambda \vdash k} \bigoplus_{a=1}^{d_\lambda} V_{(\lambda, a)}, \quad V_{(\lambda, a)} = \text{Im}(E_a^\lambda), \quad (5)$$

where the number  $d_\lambda$  stands for the total number of the standard Young tableaux, which can be constructed for a given partition  $\lambda$ .

Since the projectors  $E_a^\lambda$  with different  $a$  are connected by invertible transformations, all spaces  $V_{(\lambda, a)}$  with fixed  $\lambda$  and different  $a$  are isomorphic. Note, that the isomorphic spaces  $V_{(\lambda, a)}$  (at a fixed  $\lambda$ ) in decomposition (5) are treated as particular embeddings of the space  $V_\lambda$  into the tensor product  $V^{\otimes k}$ . Hereafter we use the notation  $V_\lambda$  for the class of the spaces  $V_{(\lambda, a)}$  equipped with one or another embedding in  $V^{\otimes k}$ .

In a similar way we define classes  $V_\mu^*$ . First, note that the Hecke symmetry being extended to the space  $(V^*)^{\otimes 2}$  is given in the basis  $x^i \otimes x^j$  as follows

$$R(x^i \otimes x^j) = R_{lk}^{ji} x^k \otimes x^l$$

(and similarly in the basis  ${}^i x \otimes {}^j x$ ). It is not difficult to see that the operator  $R$  so defined in the space  $(V^*)^{\otimes 2}$  is a Hecke symmetry. Thus, by using the above method we can introduce spaces  $V_{(\mu, a)}^*$  looking like those from (5) and define the classes  $V_\mu^*$ .

Now, let us define a rigid quasitensor Schur-Weyl category  $\text{SW}(V)$  whose objects are spaces  $V_\lambda$  and  $V_\mu^*$  labelled by partitions of nonnegative integers, as well as their tensor products  $V_\lambda \otimes V_\mu^*$  and all finite sums of these spaces.

Among the morphisms of the category  $\text{SW}(V)$  are the above left and right pairings and the set of braidings  $R_{U, W} : U \otimes W \rightarrow W \otimes U$  for any pair of objects  $U$  and  $W$ . These braidings can be defined in a natural way. In order to define them on a couple



of objects of the form  $V_\lambda \otimes V_\mu^*$  we embed them into appropriate products  $V^{\otimes k} \otimes (V^*)^{\otimes l}$  and define the braiding  $R_{U,W}$  as an appropriate restriction. Note, that all these braidings are  $R$ -invariant maps (cf. [GPS] for detail). Note that the category  $\text{SW}(V)$  is monoidal quasitensor rigid according to the standard terminology (cf. [CP]).

Now we are aiming at introducing modified reflection equation algebra and equipping objects of the category  $\text{SW}(V)$  with a structure of its modules.

Again, consider the space  $\text{End}(V)$  equipped with the basis  $l_i^j$  (see Section 2). Note that the element  $l_i^j$  acts on the elements of the space  $V$  as follows

$$l_i^j(x_k) := x_i \langle x^j, x_k \rangle = x_i B_k^j. \quad (6)$$

Introduce the  $N \times N$  matrix  $L = \|l_i^j\|$ . Also, define its "copies" by the iterative rule

$$L_{\overline{1}} := L_1 := L \otimes I, \quad L_{\overline{k+1}} := R_k L_{\overline{k}} R_k^{-1}. \quad (7)$$

Observe that the isolated spaces  $L_{\overline{k}}$  have no meaning (except for that  $L_{\overline{1}}$ ). They can be only correctly understood in the products  $L_{\overline{1}} L_{\overline{2}}$ ,  $L_{\overline{1}} L_{\overline{2}} L_{\overline{3}}$  and so on, but this notation is useful in what follows.

**Definition 6** The associative algebra generated by the unit element  $e_{\mathcal{L}}$  and the indeterminates  $l_i^j$   $1 \leq i, j \leq N$  subject to the following matrix relation

$$R_{12} L_1 R_{12} L_1 - L_1 R_{12} L_1 R_{12} - \hbar (R_{12} L_1 - L_1 R_{12}) = 0, \quad (8)$$

is called the modified reflection equation algebra (mREA) and denoted  $\mathcal{L}(R_q, \hbar)$ .

Note, that at  $\hbar = 0$  the above algebra is known as the reflection equation algebra  $\mathcal{L}(R_q)$ . Actually, at  $q \neq \pm 1$  one has  $\mathcal{L}(R_q, \hbar) \cong \mathcal{L}(R_q)$ . Since at  $\hbar \neq 0$  it is always possible to renormalize generators  $L \mapsto \hbar L$ . So, below we consider the case  $\hbar = 1$ .

Thus, the mREA is the quotient algebra of the free tensor algebra  $T(\text{End}(V))$  over the two-sided ideal, generated by the matrix elements of the left hand side of (8). It can be shown, that the relations (8) are  $R$ -invariant, that is the above two-sided ideal is invariant when commuting with any object  $U$  under the action of the braidings  $R_{U, \text{End}(V)}$  or  $R_{\text{End}(V), U}$  of the category  $\text{SW}(V)$ .

Taking into account (2) one can easily prove, that the action (6) gives a basic (vector) representation of the mREA  $\mathcal{L}(R_q, 1)$  in the space  $V$

$$\rho_1(l_i^j) \triangleright x_k = x_i B_k^j, \quad (9)$$

where the symbol  $\triangleright$  stands for the (left) action of a linear operator onto an element. Since  $B$  is non-degenerated, the representation is irreducible.

Another basic (covector) representation  $\rho_1^* : \mathcal{L}(R_q, 1) \rightarrow \text{End}(V^*)$  is given by

$$\rho_1^*(l_i^j) \triangleright x^k = -x^r R_{ri}^{kj}. \quad (10)$$

one can prove, that the maps  $\text{End}(V) \rightarrow \text{End}(V)$  and  $\text{End}(V) \rightarrow \text{End}(V^*)$  generated by

$$l_i^j \mapsto \rho_1(l_i^j) \quad \text{and} \quad l_i^j \mapsto \rho_1^*(l_i^j)$$

are the morphisms of the category  $\text{SW}(V)$ .

**Definition 7** A representation  $\rho : \mathcal{L}(R_q, 1) \rightarrow \text{End}(U)$  where  $U$  is an object of the category  $\text{SW}(V)$  is called *equivariant* if its restriction to  $\text{End}(V)$  is a categorical morphism.

Thus, the above representations  $\rho_1$  and  $\rho_1^*$  are equivariant.

Note that there are known representations of the mREA which are not equivariant. However, the class of equivariant representations of the mREA is very important. In particular, because the tensor product of two equivariant  $\mathcal{L}(R_q, 1)$ -modules can be also equipped with a structure of an equivariant  $\mathcal{L}(R_q, 1)$ -module via a "braided bialgebra structure" of the mREA.

Let us briefly describe this structure. It consists of two maps: the braided coproduct  $\Delta$  and counit  $\varepsilon$ .

The coproduct  $\Delta$  is an algebra homomorphism of  $\mathcal{L}(R_q, 1)$  into the associative algebra  $\mathbf{L}(R_q)$  which is defined as follows.

- As a vector space over the field  $\mathbb{K}$  the algebra  $\mathbf{L}(R_q)$  is isomorphic to the tensor product of two copies of mREA

$$\mathbf{L}(R_q) = \mathcal{L}(R_q, 1) \otimes \mathcal{L}(R_q, 1).$$

- The product  $\star : (\mathbf{L}(R_q))^{\otimes 2} \rightarrow \mathbf{L}(R_q)$  is defined by the rule

$$(a_1 \otimes b_1) \star (a_2 \otimes b_2) := a_1 a'_2 \otimes b'_1 b_2, \quad a_i \otimes b_i \in \mathbf{L}(R_q), \quad (11)$$

where  $a_1 a'_2$  and  $b_1 b'_2$  are the usual product of mREA elements, while  $a'_1$  and  $b'_1$  result from the action of the braiding  $R_{\text{End}(V)}$  (see Section 2) on the tensor product  $b_1 \otimes a_2$

$$a'_2 \otimes b'_1 := R_{\text{End}(V)}(b_1 \otimes a_2). \quad (12)$$

The braided coproduct  $\Delta$  is now defined as a linear map  $\Delta : \mathcal{L}(R_q, 1) \rightarrow \mathbf{L}(R_q)$  with the following properties:

$$\begin{aligned} \Delta(e_{\mathcal{L}}) &:= e_{\mathcal{L}} \otimes e_{\mathcal{L}} \\ \Delta(l_i^j) &:= l_i^j \otimes e_{\mathcal{L}} + e_{\mathcal{L}} \otimes l_i^j - (q - q^{-1}) \sum_k l_i^k \otimes l_k^j \\ \Delta(ab) &:= \Delta(a) \star \Delta(b) \quad \forall a, b \in \mathcal{L}(R_q, 1). \end{aligned} \quad (13)$$

In addition to (13), we introduce a linear map  $\varepsilon : \mathcal{L}(R_q, 1) \rightarrow \mathbb{K}$

$$\begin{aligned} \varepsilon(e_{\mathcal{L}}) &:= 1 \\ \varepsilon(l_i^j) &:= 0 \\ \varepsilon(ab) &:= \varepsilon(a)\varepsilon(b) \quad \forall a, b \in \mathcal{L}(R_q, 1). \end{aligned} \quad (14)$$

One can show (cf. [GPS]) that the maps  $\Delta$  and  $\varepsilon$  are indeed algebra homomorphisms and that they satisfy the relation

$$(\text{id} \otimes \varepsilon) \Delta = \text{id} = (\varepsilon \otimes \text{id}) \Delta.$$

Let now  $U$  and  $W$  be two equivariant mREA-modules with representations  $\rho_U : \mathcal{L}(R_q, 1) \rightarrow \text{End}(U)$  and  $\rho_W : \mathcal{L}(R_q, 1) \rightarrow \text{End}(W)$  respectively. Consider the map  $\rho_{U \otimes W} : \mathbf{L}(R_q) \rightarrow \text{End}(U \otimes W)$  defined as follows

$$\rho_{U \otimes W}(a \otimes b) \triangleright (u \otimes w) = (\rho_U(a) \triangleright u') \otimes (\rho_W(b') \triangleright w), \quad a \otimes b \in \mathbf{L}(R_q), \quad (15)$$

where

$$u' \otimes b' := R_{\text{End}(V), U}(b \otimes u).$$

Definition (15) is self-consistent since the map  $b \mapsto \rho_W(b')$  is also a representation of the mREA  $\mathcal{L}(R_q, 1)$ .

The following proposition holds true.

**Proposition 8** ([GPS]) *The action (15) defines a representation of the algebra  $\mathbf{L}(R_q)$ .*

Note again, that the equivariance of the representations in question plays a decisive role in the proof of the above proposition.

As an immediate corollary of the proposition 8 we get the rule of tensor multiplication of equivariant  $\mathcal{L}(R_q, 1)$ -modules.

**Corollary 9** *Let  $U$  and  $W$  be two  $\mathcal{L}(R_q, 1)$ -modules with equivariant representations  $\rho_U$  and  $\rho_W$ . Then the map  $\mathcal{L}(R_q, 1) \rightarrow \text{End}(U \otimes W)$  given by the rule*

$$a \mapsto \rho_{U \otimes W}(\Delta(a)), \quad \forall a \in \mathcal{L}(R_q, 1) \quad (16)$$

*is an equivariant representation. Here the coproduct  $\Delta$  and the map  $\rho_{U \otimes W}$  are given respectively by formulae (13) and (15).*

Thus, by using (16) we can extend the basic representations  $\rho_1$  and  $\rho_1^*$  to the representations  $\rho_k$  and  $\rho_l^*$  in tensor products  $V^{\otimes k}$  and  $(V^*)^{\otimes l}$  respectively. These representations are reducible, and their restrictions on the representations  $\rho_{\lambda, a}$  in the invariant subspaces  $V_{(\lambda, a)}$  (see (5)) are given by the projections

$$\rho_{\lambda, a} = E_a^\lambda \circ \rho_k \quad (17)$$

and similarly for the subspaces  $V_{(\mu, a)}^*$ . By using (16) once more we can equip each object of the category  $\text{SW}(V)$  with the structure of an equivariant  $\mathcal{L}(R_q, 1)$ -module.

## 5 Quantum Lie algebras related to Hecke symmetries

In this section we consider the question to which extent one can use the scheme of section 2 in the case of non-involutive Hecke symmetry  $R$  for definition of the corresponding Lie algebra-like object. For such an object related to a Hecke symmetry  $R$  we use the term *quantum* or *braided* Lie algebra. Besides, we require the mREA, connected with the same symmetry  $R$ , to be an analog of the enveloping algebra of the quantum Lie algebra. Finally, we compare the properties of the above generalized Lie algebras and quantum ones.

Let us recall the interrelation of a usual Lie algebra  $\mathfrak{g}$  and its universal enveloping algebra  $U(\mathfrak{g})$ . As is known, the universal enveloping algebra for a Lie algebra  $\mathfrak{g}$  is a unital associative algebra  $U(\mathfrak{g})$  possessing the following properties:

- There exists a linear map  $\tau : \mathfrak{g} \rightarrow U(\mathfrak{g})$  such that 1 and  $\text{Im } \tau$  generate the whole  $U(\mathfrak{g})$ .
- The Lie bracket  $[x, y]$  of any two elements of  $\mathfrak{g}$  has the image

$$\tau([x, y]) = \tau(x)\tau(y) - \tau(y)\tau(x).$$

Let us rewrite these formulae in an equivalent form. Note that the tensor square  $\mathfrak{g} \otimes \mathfrak{g}$  splits into the direct sum of symmetric and skew symmetric components

$$\mathfrak{g} \otimes \mathfrak{g} = \mathfrak{g}_s \oplus \mathfrak{g}_a, \quad \mathfrak{g}_s = \text{Im } \mathcal{S}, \quad \mathfrak{g}_a = \text{Im } \mathcal{A},$$

where  $\mathcal{S}$  and  $\mathcal{A}$  are the standard (skew)symmetrizing operators

$$\mathcal{S}(x \otimes y) = x \otimes y + y \otimes x, \quad \mathcal{A}(x \otimes y) = x \otimes y - y \otimes x,$$

where we neglect the usual normalizing factor 1/2. Then the skew-symmetry property of the classical Lie bracket is equivalent to the requirement

$$[\ , \ ] \mathcal{S}(x \otimes y) = 0. \quad (18)$$

The image of the bracket in  $U(\mathfrak{g})$  is presented as follows

$$\tau([x, y]) = \circ \mathcal{A}(\tau(x) \otimes \tau(y)), \quad (19)$$

where  $\circ$  stands for the product in the associative algebra  $U(\mathfrak{g})$ .

- The Jacobi identity for the Lie bracket  $[\ , \ ]$  translates into the requirement that the correspondence  $x \mapsto [x, \ ]$  generate the (adjoint) representation of  $U(\mathfrak{g})$  in the linear subspace  $\tau(\mathfrak{g}) \subset U(\mathfrak{g})$ .

So, we define a braided Lie algebra as a linear subspace  $\mathcal{L}_1 = \text{End}(V)$  of the mREA  $\mathcal{L}(R_q, 1)$ , which generates the whole algebra and is equipped with the quantum Lie bracket. We want the bracket to satisfy some skew-symmetry condition, generalizing (18), and define a representation of the mREA in the same linear subspace  $\mathcal{L}_1$  via an analog of the Jacobi identity.

As  $\mathcal{L}_1$ , let us take the linear span of mREA generators

$$\mathcal{L}_1 = \text{End}(V) \cong V \otimes V^*.$$

Together with the unit element this subspace generate the whole  $\mathcal{L}(R_q, 1)$  by definition.

In order to find the quantum Lie bracket, consider a particular representation of  $\mathcal{L}(R_q, 1)$  in the space  $\text{End}(V)$ . In this case the general formula (16) reads

$$l_i^j \mapsto \rho_{V \otimes V^*}(\Delta(l_i^j)),$$

where we should take the basic representations (9) and (10) as  $\rho_V(l_i^j)$  and  $\rho_{V^*}(l_i^j)$  respectively. Omitting straightforward calculations, we write the final result in the compact matrix form

$$\rho_{V \otimes V^*}(L_{\overline{1}}) \triangleright L_{\overline{2}} = L_1 R_{12} - R_{12} L_1, \quad (20)$$

where the matrix  $L_{\overline{k}}$  is defined in (7).

Let us define

$$[L_{\overline{1}}, L_{\overline{2}}] = L_1 R_{12} - R_{12} L_1. \quad (21)$$

The generalized skew-symmetry (the axiom 1 from Section 2) of this bracket is now modified as follows. In the space  $\mathcal{L}_1 \otimes \mathcal{L}_1$  one can construct two projection operators  $\mathcal{S}_q$  and  $\mathcal{A}_q$  which are interpreted as  $q$ -symmetrizer and  $q$ -skew-symmetrizer respectively (cf. [GPS]). Then straightforward calculations show that the above bracket satisfies the relation

$$[\ , \ ] \mathcal{S}_q(L_{\overline{1}} \otimes L_{\overline{2}}) = 0, \quad (22)$$

which is the generalized skew-symmetry condition, analogous to (18).

Moreover, if we rewrite the defining commutation relations of the mREA (8) in the equivalent form

$$L_{\overline{1}} L_{\overline{2}} - R_{12}^{-1} L_{\overline{1}} L_{\overline{2}} R_{12} = L_1 R_{12} - R_{12} L_1, \quad (23)$$

we come to a generalization of the formula (19).

By introducing an operator

$$Q : \mathcal{L}_1^{\otimes 2} \rightarrow \mathcal{L}_1^{\otimes 2}, \quad Q(L_{\overline{1}} L_{\overline{2}}) = R_{12}^{-1} L_{\overline{1}} L_{\overline{2}} R_{12}$$

we can present the relation (23) as follows

$$L_{\overline{1}} L_{\overline{2}} - Q(L_{\overline{1}} L_{\overline{2}}) = [L_{\overline{1}}, L_{\overline{2}}]. \quad (24)$$

It looks like the defining relation of the enveloping algebra of a generalized Lie algebra. (Though we prefer to use the notations  $L_{\overline{k}}$  it is possible to exhibit the maps  $Q$  and  $[\ , \ ]$  in the basis  $l_i^j \otimes l_k^m$ .) Observe that the map  $Q$  is a braiding. Also, note that the operators  $\mathcal{S}_q$  and  $\mathcal{A}_q$  can be expressed in terms of  $Q$  and its inverse (cf. [GPS]).

We call the data  $(\mathfrak{g} = \mathcal{L}_1, Q, [\ , \ ])$  the  $gl$  type quantum (braided) Lie algebra. Note that if  $q = 1$  (i.e. the symmetry  $R$  is involutive) then  $Q = R_{\text{End}(V)}$  and this quantum Lie algebra is nothing but the generalized Lie algebra  $gl(V_R)$  and the corresponding mREA becomes isomorphic to its enveloping algebra.

Let us list the properties of the the quantum Lie algebra in question.

- The bracket  $[\ , \ ]$  is skew-symmetric in the sense of (22).
- The  $q$ -Jacobi identity is valid in the following form

$$[\ , \ ] [\ , \ ]_{12} = [\ , \ ] [\ , \ ]_{23} (I - Q_{12}). \quad (25)$$

- The bracket  $[\ , \ ]$  is  $R$ -invariant. Essentially, this means that the following relations hold

$$\begin{aligned} R_{\text{End}(V)} [\ , \ ]_{23} &= [\ , \ ]_{12} (R_{\text{End}(V)})_{23} (R_{\text{End}(V)})_{12}, \\ R_{\text{End}(V)} [\ , \ ]_{12} &= [\ , \ ]_{23} (R_{\text{End}(V)})_{12} (R_{\text{End}(V)})_{23}. \end{aligned} \quad (26)$$

So, the adjoint action

$$L_{\overline{1}} \triangleright L_{\overline{2}} = [L_{\overline{1}}, L_{\overline{2}}]$$

is indeed a representation. By chance (!) the representation  $\rho_{V \otimes V^*}$  coincides with this adjoint action.

Turn now to the question of the "sl-reduction", that is, the passing from the mREA  $\mathcal{L}(R_q, 1)$  to the quotient algebra

$$\mathcal{SL}(R_q) := \mathcal{L}(R_q, 1) / \langle \text{Tr}_R L \rangle, \quad \text{Tr}_R L := \text{Tr}(CL), \quad (27)$$

(see Section 2 for the operator  $C$ ). The element  $\ell := \text{Tr}_R L$  is central in the mREA, which can be easily proved by calculating the  $R$ -trace in the second space of the matrix relation (8).

To describe the quotient algebra  $\mathcal{SL}(R_q)$  explicitly, we pass to a new set of generators  $\{f_i^j, \ell\}$ , connected with the initial one by a linear transformation:

$$l_i^j = f_i^j + (\text{Tr}(C))^{-1} \delta_i^j \ell \quad \text{or} \quad L = F + (\text{Tr}(C))^{-1} I \ell, \quad (28)$$

where  $F = \|f_i^j\|$ . Hereafter we assume that  $\text{Tr} C = \ell_i^i \neq 0$ . (So, the Lie super-algebras  $gl(m|m)$  and their  $q$ -deformations are forbidden.) Obviously,  $\text{Tr}_R F = 0$ , i.e. the generators  $f_i^j$  are dependent.

In terms of the new generators, the commutation relations of the mREA read

$$\begin{cases} R_{12} F_1 R_{12} F_1 - F_1 R_{12} F_1 R_{12} = (e_{\mathcal{L}} - \frac{\omega}{\text{Tr}(C)} \ell) (R_{12} F_1 - F_1 R_{12}) \\ \ell F = F \ell, \end{cases}$$

where  $\omega = q - q^{-1}$ . Now, it is easy to describe the quotient (27). The matrix  $F = \|f_i^j\|$  of  $\mathcal{SL}(R_q)$  generators satisfy the same commutation relations (8) as the matrix  $L$

$$R_{12} F_1 R_{12} F_1 - F_1 R_{12} F_1 R_{12} = R_{12} F_1 - F_1 R_{12}, \quad (29)$$

but the generators  $f_i^j$  are linearly dependent.

Rewriting this relation in the form similar to (24) we can introduce an  $sl$ -type bracket. However for such a bracket the  $q$ -Jacobi identity fails. This is due to the fact the element  $\ell$  comes in the relations for  $f_i^j$  (at  $q = 1$  this effect disappears). Nevertheless, we can construct a representation

$$\rho_{V \otimes V^*} : \mathcal{SL}(R_q) \rightarrow \text{End}(V \otimes V^*)$$

which is an analog of the adjoint representation. In order to do so, we rewrite the representation (20) in terms of the generators  $f_i^j$  and  $\ell$ . Taking relation (28) into account, we find, after a short calculation

$$\begin{aligned} \rho_{V \otimes V^*}(\ell) \triangleright \ell &= 0, & \rho_{V \otimes V^*}(F_1) \triangleright \ell &= 0, \\ \rho_{V \otimes V^*}(\ell) \triangleright F_1 &= -\omega \text{Tr}(C) F_1 \\ \rho_{V \otimes V^*}(F_1) \triangleright F_2 &= F_1 R_{12} - R_{12} F_1 + \omega R_{12} F_1 R_{12}^{-1}. \end{aligned} \quad (30)$$

Namely, the last formula from this list defines the representation  $\rho_{V \otimes V^*}$ . However, in contrast with the mREA  $\mathcal{L}(R_q, 1)$ , this map is different from that defined by the bracket  $[\cdot, \cdot]$  reduced to the space  $\text{span}(f_i^j)$ . This is reason why the " $q$ -adjoint" representation cannot be presented in the form (25). (Also, note that though  $\ell$  is central it acts in a non-trivial way on the elements  $f_i^j$ .)

Moreover, any object  $U$  of the category  $\text{SW}(\mathbb{V})$  above such that

$$\rho_U(\ell) = \chi I_U, \quad \chi \in \mathbb{K}$$

is a scalar operator, can be equipped with an  $\mathcal{SL}(R_q)$ -module structure. First, let us observe that for any representation  $\rho_U : \mathcal{L}(R_q, 1) \rightarrow \text{End}(U)$  and for any  $z \in \mathbb{K}$  the map

$$\rho_U^z : \mathcal{L}(R_q, 1) \rightarrow \text{End}(U), \quad \rho_U^z(l_i^j) = z\rho_U(l_i^j) + \delta_i^j(1-z)(q - q^{-1})^{-1} I_U$$

is a representation of this algebra as well.

By using this freedom we can convert a given representation  $\rho_U : \mathcal{L}(R_q, 1) \rightarrow \text{End}(U)$  with the above property into that  $\rho_U^z$  such that  $\rho_U^z(\ell) = 0$ . Thus we get a representation of the algebra  $\mathcal{SL}(R_q)$ . Explicitly, this representation is given by the formula

$$\tilde{\rho}(f_i^j) = \frac{1}{\xi} \left( \rho(l_i^j) - (\text{Tr}(C))^{-1} \rho(\ell) \delta_i^j \right), \quad \xi = 1 - (q - q^{-1})(\text{Tr}(C))^{-1} \chi. \quad (31)$$

The data  $(\text{span}(f_i^j), Q, [ , ])$  where the bracket stands for the l.h.s. of (24) restricted to  $\text{span}(f_i^j)$  is called the *sl*-type quantum (braided) Lie algebra.

Note that in the particular case related to the QG  $U_q(\mathfrak{sl}(n))$  this quantum algebra can be treated in terms of [LS] where an axiomatic approach to the corresponding Lie algebra-like object is given. However, we think that any general axiomatic definition of such objects is somewhat useless (unless the corresponding symmetry is involutive). Our viewpoint is motivated by the fact that for  $B_n$ ,  $C_n$ ,  $D_n$  series there do not exist "quantum Lie algebras" such that their enveloping algebras have good deformation properties. As for the  $A_n$  series (or more generally, for any skew-invertible Hecke symmetry) such objects exist and can be explicitly exhibited via the mREA. Their properties differ from those listed in [W, GM] in the framework of an axiomatic approach to Lie algebra-like objects.

Completing the paper, we want to emphasize that the above coproduct can be useful for definition of a "braided (co)adjoint vector field". In the  $\mathcal{L}(R_q, 1)$  case these fields are naturally introduced through the above adjoint action extended to the symmetric algebra of the space  $\mathcal{L}_1$  by means of this coproduct. The symmetric algebra can be defined via the above operators  $\mathcal{S}_q$  and  $\mathcal{A}_q$ . In the  $\mathcal{SL}(R_q)$  case a similar treatment is possible if  $\text{Tr } C \neq 0$ .

## References

- [CP] Chari V., Pressley A. *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994.
- [DGG] Delius G.W., Gardner C., Gould M.D. *The structure of quantum Lie algebra for the classical series  $B_l$ ,  $C_l$  and  $D_l$* , J.Phys.A: Math. Gen. **31** (1998) 1995–2019.
- [GM] Gomez X., Majid S. *Braided Lie algebras and bicovariant differential calculi over co-quasitriangular Hopf algebras*, J. of Algebra, **261** (2003) 334–388.
- [G1] Gurevich D. *Generalized translation operators on Lie groups*, Engl. transl.: Soviet J. Contemporary Math. Anal. **18** (1983) 57–90.

- [G2] Gurevich D. *Algebraic aspects of the Yang-Baxter equation*, English translation: Leningrad Math. **2** (1991) 801 – 828.
- [GPS] Gurevich D., Pyatov P., Saponov P. *Representations theory of (modified) Reflection Equation Algebra of  $GL(m|n)$  type*, St. Petersburg Math. J., submitted.
- [LS] Lyubashenko V., Sudbery A. *Generalized Lie algebras of type  $A_n$* , J. Math. Phys. **39** (1998) 3487-3504.
- [PP] Polishchuk A., Positselski L. *Quadratic Algebras*, University Lecture Series, **37**, American Mathematical Society, Providence, Rhode Island.
- [Sh] Scheunert M. *Generalized Lie algebra*, J. Math. Phys. **20** (1979) 712-720.
- [W] Woronowicz S. *Differential Calculus on Compact Matrix Pseudogroups (Quantum groups)*, CMP **122** (1989) 125–170.